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Simplification of the Hamilton-Jacobi Functional of General Relativity†

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Abstract

In follow-up of an earlier paper by Komar it is shown that the Lagrangian of general relativity can be chosen so that $S = \int p^{mn} g_{mn} \, d^3x$. This result holds without the requirement of special boundary conditions.

In an earlier paper one of us (A.K.) had shown that Dirac's choice of Lagrangian (Dirac, 1958, 1959) leads to an expression for the Hamilton-Jacobi functional that, except for a term that is functionally independent of the g_{mn}, is homogeneous in the g_{mn} of the first degree (Komar, 1970). It was conjectured that the extra term might indicate the existence of a topological invariant. In this note we shall show that the extra term, being a three-dimensional divergence, can be made to vanish with an appropriate choice of Lagrangian that differs from that of Dirac only by a three-dimensional divergence. Dirac's and our Lagrangians lead to the same expression for the canonical momentum densities (the p^{mn}) and the total Hamiltonian. (Our Hamiltonian density differs from that of Dirac merely by a spatial divergence.)

We adopt as our Lagrangian density the following expression:

$$L = \sqrt{(-g)}\, R + (\mathscr{S}\delta_0{}^\rho)_{,\rho} \tag{1}$$

$$\mathscr{S} = \sqrt{-g}\left[g^{\kappa\lambda}\begin{Bmatrix} 0 \\ \kappa\lambda \end{Bmatrix} - g^{0\kappa}\begin{Bmatrix} \rho \\ \kappa\rho \end{Bmatrix} - g^{00}\left(\frac{g^{0\kappa}}{g^{00}}\right)_{,\kappa}\right]$$

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The resulting expressions for the canonical momentum densities are:

$$p^{mn} = \sqrt{-g}(e^{mn} e^{ab} - e^{ma} e^{nb}) \left\{ \begin{matrix} 0 \\ ab \end{matrix} \right\} \tag{2}$$

The notation adopted in equations (1) and (2) is the following: g is the four-dimensional metric determinant, $g^{\kappa\lambda}$ are the components of the four-dimensional contravariant metric tensor, and e^{mn} represent the elements of the matrix inverse to the 3×3 matrix g_{mn}, just as in Dirac's papers,

$$e^{mn} = g^{mn} - \frac{g^{m0} g^{n0}}{g^{00}} \tag{3}$$

When the field equations are satisfied, the action functional becomes simply:

$$S = \int \mathscr{S} d^3 x \tag{4}$$

defined on a space-like three-dimensional hypersurface. This is because the four-dimensional curvature scalar R vanishes on account of Einstein's field equations. By a straightforward computation one can show that the expression (4), with $\mathscr{S}$ taken from (1), equals the integral

$$I = \int p^{mn} g_{mn} d^3 x \tag{5}$$

taken over the same domain of integration. In view of the fact that in the Hamilton-Jacobi theory the canonical momentum densities are the variational derivatives of S with respect to the respective configuration variables, in our case the g_{mn}, the first-degree homogeneity of S with respect to the g_{mn} follows immediately. Incidentally, the two integrals (4) and (5) can both be cast into the simple form:

$$S = I = -2 \int \sqrt{g^{00}} \left(\frac{\sqrt{(-g)} g^{0\rho}}{\sqrt{g^{00}}} \right)_{,\rho} d^3 x \tag{6}$$

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